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On a Banach Algebra of Varopoulos

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0. INTRODUCTION

K_1 and K_2 are compact metric spaces. The tensor algebra

$$V = V(K) = C(K_1) \hat{\otimes} C(K_2)$$

is the Banach algebra of all functions f continuous on the product $K = K_1 \times K_2$ which have a representation $f = \sum_1^\infty g_j \otimes h_j$ with $g_j \in C(K_1)$, $h_j \in C(K_2)$ for $j = 1, 2, \ldots$ such that $\sum_1^\infty \| g_j \|_\infty \| h_j \|_\infty < \infty$. The norm on V is the infimum of the numbers $\sum_1^\infty \| g_j \|_\infty \| h_j \|_\infty$ such that $f = \sum_1^\infty g_j \otimes h_j$. It is easily seen that the maximal ideal space of V is K. The algebra $\tilde{V}$ consists of those functions f that are uniform limits (on K) of sequences bounded in V. The norm in $\tilde{V}$ is given by

$$\| f \|_{\tilde{V}} = \inf\{\sup_j(\| f_j \|_V : f_j \in V) : \lim \| f_j - f \|_\infty = 0\}.$$

We study here the relationship between $\tilde{V}$ and its subalgebra V, and particularly the properties of the natural projection π from the maximal ideal space M of $\tilde{V}$ onto the maximal ideal space K of V. A set $\pi^{-1}(x)$ for $x \in K$ will be called a *fiber*.

Under the further assumption that both K_1 and K_2 are infinite, we have the following results:

THEOREM 1. *There exists $g \in \tilde{V}$ such that $g \notin V$, but $g^n \in V$ for $n \geqslant 2$.*

THEOREM 2. *Each fiber $\pi^{-1}(x)$ is connected.*

THEOREM 3. *If χ_j is a sequence in M which converges in the Gelfand topology to a maximal ideal χ of $\tilde{V}$, then either $\chi \in K$, or $\pi(\chi_j)$ is eventually constant.*

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THEOREM 4. *If a fiber $\pi^{-1}(x)$ contains more than one point, then it has non-empty interior in the maximal ideal space M.*

THEOREM 5. *If F is a complex-valued function defined on $[-1, 1]$ and $F \circ f \in \tilde{V}$ whenever $f \in \tilde{V}$ and $f(K) \subseteq [-1, 1]$, then F is the restriction to $[-1, 1]$ of an entire function.*

In each of Sections 1–5 we prove a theorem. In Section 6 we list some unanswered question. In the remainder of this section we make some remarks illustrating the local character of V and $\tilde{V}$, state corollaries to our theorems, and give several examples of the algebras V and $\tilde{V}$.

Remarks. (i) It is easily seen, using a partition of unity of the form $g_j \otimes h_j$, that a function $f \in C(K)$ belongs to V (or $\tilde{V}$) if it agrees in a neighborhood of each point $x \in K$ with a function belonging to V (or $\tilde{V}$).

(ii) If an element f of $\tilde{V}$ is constant in a neighborhood of a point $x \in K$, then f is *constant on the fiber* $\pi^{-1}(x)$, that is, if $\chi \in \pi^{-1}(x)$, then $\chi(f) = f(x)$.

COROLLARY TO THEOREM 1. *The subalgebra V' of $\tilde{V}$ consisting of all $f \in \tilde{V}$ which are constant on fibers is strictly larger than V. The maximal ideal space of V' is K.* (Compare with [9].)

Proof. The only statement which is not immediate is the last: If I is a maximal ideal of V' which is not the ideal $I(x)$ of all $f \in V'$ which vanish at $x \in K$, then there is an element $g \in I$ such that $g(x) \neq 0$. Since I is an ideal, $\bar{g}g \in I$. If $I \neq I(x)$ for all $x \in K$, then there is an $h \in I$ such that h is bounded away from zero on K. Since h is constant on fibers, $h^{-1} \in \tilde{V}$. Clearly $h^{-1} \in V'$, so $I = V'$.

COROLLARY TO THEOREMS 2 AND 3. *The only continuous cross-section for π is the identity map $K \to K \subseteq M$.*

Proof. Immediate from either Theorem.

EXAMPLES. (i) K_1 and K_2 are finite. Then $V = C(K_1 \times K_2) = \tilde{V}$. The norms of V and of $C(K_1 \times K_2)$, while equivalent, are not equal [*10*, p. 82 ff].

(ii) K_1 infinite, K_2 finite. Then each point in K has an open neighborhood of the form $K_1 \times \{z\}$, where $z \in K_2$. Using Remark (i) above, we see that $V = C(K_1 \times K_2)$.

(iii) $K_j = \{0, \frac{1}{2},..., 1/n,...\}$ for $j = 1, 2$. Then

$$V \neq \tilde{V} \neq C(K_1 \times K_2).$$

But from Example (ii) we see that if W is any neighborhood of $(0, 0)$, then $C(K_1 \times K_2 \setminus W) \subsetneq V$. Hence the only nontrivial fiber is $\pi^{-1}((0, 0))$. The maximal ideal space M of this $\tilde{V}$ is the union of K and $\pi^{-1}((0, 0))$, where $K = K_1 \times K_2$.

The nontrivial fiber $\pi^{-1}((0, 0))$ has the property that

$$\pi^{-1}((0,0)) \setminus \{(0,0)\}$$

is disconnected. To see this, let L be the "even" subset of K_1, that is $L = \{\frac{1}{2}, \frac{1}{4},...\}$ and let $L' = \{\frac{1}{3}, \frac{1}{5},...\}$. Let $\tilde{V}_0$ be the ideal of those $f \in \tilde{V}$ such that $f((0, 0)) = 0$. The maximal ideal space of $\tilde{V}_0$ is $M \setminus \{(0, 0)\}$. Clearly $\tilde{V}_0$ is the sum of the ideals I and I' consisting of those functions which vanish off $L \times K_2$ and $L' \times K_2$. Hence the maximal ideal space of $\tilde{V}_0$ is the union of the maximal ideal spaces of these two ideals, that is, of two open disjoint sets. Any homeomorphism of K_1 which maps L onto L', and L' onto L induces an automorphism of $\tilde{V}$ which maps I isometrically onto I'. Hence the maximal ideal spaces of I and I' are homeomorphic and have nonvoid intersection with $\pi^{-1}((0, 0))$. Hence the set $\pi^{-1}((0, 0)) \setminus \{(0, 0)\}$ is disconnected.

(iv) K_1 and K_2 are compact metric groups. Then the group action induces homeomorphisms of the maximal ideal space M of $\tilde{V}$. Given any two fibers, there is obviously one such homeomorphism which maps one fiber onto the other. Hence M is, as a set, the Cartesian product of the spaces K and $\pi^{-1}(x)$. The corollary to Theorems 2 and 3 shows this product is not topological.

(v) In certain cases V is a restriction algebra $A(E)$ of Fourier transforms [*10*, p. *75*]. Then $\tilde{V}$ is the algebra $B(E)$ of Katznelson and McGehee [*5*].

Theorems 1 and 5 have direct analogs with results in Fourier analysis (see [*3*], [*9*], and [*7*, Section 6.3], respectively). Theorems 2, 3, and 4 have direct analogs with results for H^∞, the algebra of bounded analytic functions on the open unit disc (see [*4*, pp. 188, 165, 165], respectively). Our proofs of Theorems 3 and 5 owe much to these previous results. Varopoulos [*12*] was the first to study $\tilde{V}$. Theorems 1–4 were announced and to some extent proved in [*2*].

1. Proof of Theorem 1

We shall find a sequence of functions $g_j \in V$ such that

$$\|g_j\|_V = 1, \qquad j = 1, 2, \ldots \tag{1}$$

$$\|g_j^2\|_V < 2^{-j}, \qquad j = 1, 2, \ldots \tag{2}$$

and such that the support of g_j is contained in $Y_j \times Z_j$, where the $Y_1, Y_2, \ldots$ are pairwise disjoint in K_1 and the $Z_1, Z_2, \ldots$ are pairwise disjoint in K_2. As we shall see, under these conditions the function $g = \sum_1^\infty g_j$ is in $\tilde{V}$ but not in V. But it is evident that $g^n \in V$ if $n \geqslant 2$.

Lemma A. *For each integer j, there exists an integer $N = N(j)$ such that if F_1 and F_2 are two discrete spaces each having N points and $F = F_1 \times F_2$, then there exists an $f_j \in V(F)$ such that*

$$\|f_j\|_{V(F)} = 1 \qquad \text{and} \qquad \|f_j^2\|_{V(F)} < 2^{-j}. \tag{3}$$

Proof of Lemma A. We use the fact that square root does not operate on $V(K')$ where K' is the product of two convergent sequences (see Example (iii) above). The proof of this fact [*11*] shows that there exists an $f \in V(K')$ such that $f(x) \geqslant 0$ for all $x \in K$, and such that the norm of $(f + a\mathbf{1})^{1/2}$ tends to infinity as $a \in]0, \infty[$ tends to 0; $\mathbf{1}$ is the identity of V. Pick such an f and an a such that

$$\|f + a\mathbf{1}\| = 1 \qquad \text{and} \qquad \|(f + a\mathbf{1})^{1/2}\| > 2^j. \tag{4}$$

Varopoulos [*12*] has shown that for this K', V- and $\tilde{V}$-norms agree on $V(K') \subseteq \tilde{V}(K')$, and that the $\tilde{V}$-norm is given by [*12*, Formula (5)]:

$$\|g\|_{\tilde{V}(K')} = \sup\{\|g|_F\|_{V(F)}\}, \tag{5}$$

where the supremum in (5) is taken over sets F which are products of finite subsets $F_j \subseteq K_j$ for $j = 1, 2$.

To find our $N = N(j)$ and f_j we simply pick N so large that for some finite subsets $F_j \subseteq K_j$ of N elements we have

$$\|(f + a\mathbf{1})^{1/2}|_F\|_{V(F)} > 2^j. \tag{6}$$

We now renormalize the function in (6):

$$f_j = [\|(f + a\mathbf{1})^{1/2}|_F\|_{V(F)}]^{-1} (f + a\mathbf{1})^{1/2}|_F.$$

Proof of Theorem 1. We suppose that $g_1, \ldots, g_n$ have been found which satisfy (1) and (2), which have $\operatorname{supp}(g_j) \subset Y_j \times Z_j$ where $Y_1, ..., Y_n$ are pairwise disjoint in K_1 and $Z_1, ..., Z_n$ are pairwise disjoint in K_2, and such that $K_1 \setminus \bigcup_1^n Y_j$ and $K_2 \setminus \bigcup_1^n Z_j$ are both infinite and open.

From the preceding conditions on the Y_j and Z_j we see that we can find closed sets Y_{n+1} and Z_{n+1} such that Y_{n+1} and Z_{n+1} each contains at least $N = N(n + 1)$ points (N from Lemma A) in its interior, and such that $Y_1, ..., Y_{n+1}$ are pairwise disjoint, $Z_1, ..., Z_{n+1}$ are pairwise disjoint, and such that $K_1 \setminus \bigcup_1^{n+1} Y_j$ and $K_2 \setminus \bigcup_1^{n+1} Z_j$ are both infinite. Let f_{n+1} be the function given by Lemma A. It is easy to see that f_{n+1} has an extension to a function $f' \in V(K)$ such that the support of f' is contained in $Y_{n+1} \times Z_{n+1}$. From the definition of the norm on $V(K)$, we see f' may be chosen so that

$$\|f'^2\|_{V(K)} < 2^{-j}.$$

If $\|f'\|_{K(V)} > 1$, we renormalize. This renormalized function is the function g_{n+1}. If we do not need to renormalize, take $g_{n+1} = f'$.

All we need show now is that the sum $g = \sum_1^\infty g_j \in \tilde{V}$ and that $g \notin V$.

A repeated application of Lemma B below shows that the norms of the finite sums $\sum_{j=1}^L g_j$ are bounded by one. Since these finite sums converge uniformly to g, $g \in \tilde{V}$.

Lemma C and (1) show that $g \notin V$; the point x of the statement of Lemma C is chosen to be an accumulation point of the supports of the g_j which does not lie in the union of those supports.

LEMMA B. [*12*, lemma 2]. *Let $U_{ij} \subseteq K_j$ be open subsets, $i, j = 1, 2$. Suppose*

$$U_{j1} \cap U_{j2} = \varnothing, \qquad j = 1, 2.$$

Let f_i be elements of $V(K)$ having support $f_i \subseteq U_{1i} \times U_{2i}$. Then $f_1 + f_2$ has $V(K)$-norm equal to the supremum of the $V(K)$-norms of the f_i.

LEMMA C. *Let $x \in K$ and let $f \in V(K)$ vanish at x. Then there exists a sequence g_j of elements of $V(K)$ such that g_j is zero in a neighborhood of x, $j = 1, 2, ...$, and such that $g_j f$ converges to f in the norm of $V(K)$.*

Proof of Lemma C. By partitioning unity into two pieces we see that it is sufficient to show that if $U(n)$ is a sequence of rectangular closed neighborhoods of x (which tend sufficiently fast to $\{x\}$), then the norm of (the restriction of) f in the restriction algebra $V(U(n))$ tends to zero. But if this last is false, there exists an $\epsilon > 0$ and μ_n in the unit ball of the dual space [6] of $V(K)$ such that μ_n is supported on $U(n)$ and such that for $f_n = f|_{U(n)}$, $|\langle f_n, \mu_n\rangle| > \epsilon$, $n = 1, 2,\ldots$

Let μ be a weak-* accumulation point of the μ_n in the unit ball of the dual space of $V(K)$. Then

$$|\langle f, \mu\rangle| \geqslant \liminf |\langle f_n, \mu_n\rangle| \geqslant \epsilon \neq 0,$$

since f agrees with f_n on the support of μ_n. But the support of μ is contained in the intersection of the $U(n)$. We may choose the $U(n)$ to have intersection $\{x\}$. Then μ must be a multiple of point mass at x. Then $f(x) = 0$ implies $\langle f, \mu\rangle = 0$. This contradiction completes the proof of the lemma.

2. Proof of Theorem 2

We use the well-known theorem of Silov [8] which states that if the maximal ideal space of a Banach algebra contains a compact open subset, then there is an idempotent in the algebra which has Gelfand transform one on the compact open subset and zero on the remainder of the maximal ideal space.

Let I be the ideal of functions in $\tilde{V}$ whose Gelfand transforms vanish on the fiber $\pi^{-1}(x)$. It is easily seen that the maximal ideal space of $\tilde{V}/I$ is $\pi^{-1}(x)$. If the fiber $\pi^{-1}(x)$ is disconnected, let $f + I$ be an idempotent given by Silov's Theorem. We shall show that f has a representative g such that $g = g^2$ holds in a neighborhood of x. From Remark (ii) of Section 0 above, we see that g, and therefore f, is constant on the fiber $\pi^{-1}(x)$. Hence the fiber is connected. It remains to find the representative g.

Suppose S is any rectangular closed neighborhood of x. Consider the three sets:

$$\begin{aligned} &\{\chi : |\chi(f)| > 1/2, \chi \in \pi^{-1}(S)\},\\ &\{\chi : |\chi(f)| < 1/2, \chi \in \pi^{-1}(S)\},\\ &\{\chi : |\chi(f)| = 1/2, \chi \in \pi^{-1}(S)\}. \end{aligned} \tag{7}$$

It is easily seen that $\pi^{-1}(S)$ is the maximal ideal space of the restriction of $\tilde{V}$ to S, that is, the quotient $\tilde{V}/J(S)$, where $J(S)$ is the ideal of $f \in \tilde{V}$ such that $\mu(f) = 0$ for all $\mu \in \pi^{-1}(S)$.

Suppose for each S, the third set is never empty. Then choose a sequence $S(n)$ of closed rectangular neighborhoods of x such that $\bigcap_{n=1}^{\infty} S(n) = \{x\}$. Let χ_n be an element of $\pi^{-1}(S(n))$ such that $|\chi_n(f)| = \frac{1}{2}$, and let χ be an accumulation point of the χ_n. Now χ must lie in the intersection of the cylinders $\pi^{-1}(S(n))$, that is in $\pi^{-1}(x)$. Hence we have a $\chi \in \pi^{-1}(x)$ such that $|\chi(f)| = \frac{1}{2}$. This contradiction shows that for one closed rectangular neighborhood S of x, the maximal ideal space $\pi^{-1}(S)$ is the union of the first two sets in (7). These two sets are open compact, so there is an idempotent $g' + J(S)$ which is one on the first and zero on the second. Pick any representative g of g'. It is easy to see that $g = g^2$ in the neighborhood S. This completes the proof of Theorem 2.

3. Proof of Theorem 3

We may assume that $\chi \notin K$ and that the χ_j are all in different fibers $\pi^{-1}(x_j)$. Because π is continuous, x_j converges to $x = \pi(\chi)$. Because $\chi \notin K$, there is an $f \in \tilde{V}$ such that $f(x) = 0$ and $\chi(f) = 1$.

We shall show that these hypotheses are contradictory by finding a subsequence $\chi_{j'}$ of χ_j which does not converge. Since the π-projections $x_j = (y_j, z_j)$ are distinct, there exists a subsequence $x_{j'} = (y_{j'}, z_{j'})$ of x_j such that either the $y_{j'}$ are distinct in K_1, or the $z_{j'}$ are distinct in K_2. We may assume the $y_{j'}$ are distinct in K_1. Let $\chi_{j'}$ be the (unique) element of the sequence χ_j which belongs to $\pi^{-1}(x_{j'})$.

The $y_{j'}$ converge to y, the first coordinate of $x = \pi(\chi)$. Choose Y_j a closed neighborhood of $y_{j'}$ such that: the Y_j are pairwise disjoint; $y \in Y_j$, all $j = 1, 2, \ldots$, and such that for each neighborhood Y of y, there exists an integer $N(Y)$ such that $j \geqslant N(Y)$ implies $Y_j \subseteq Y$.

Choose $k_j \in C(K_1)$ such that $k_j(y_{j'}) = 1$ and support of k_j is contained in Y_j, and $\| k_j \|_\infty \leqslant 1$. Consider the finite sums

$$F_n = \Big(\sum_{j=1}^{n} (-1)^j k_j \otimes 1\Big)(f). \tag{8}$$

Since the sum $\sum_1^n (-1)^j k_j$ has supremum one on K_1, the $\tilde{V}$-norm of F_n is bounded by $\|f\|$. For $n \geqslant j$, F_n agrees on a neighborhood of $x_{j'}$ with $(-1)^j k_j \otimes 1(f)$. Hence by Remark (ii) of Section 0 above and the fact that $k_j \otimes 1$, being in V, is constant on fibers, we have:

$$\chi_{j'}(F_n) = (-1)^j k_j(y_{j'}) \chi_{j'}(f) = (-1)^j \chi_{j'}(f). \tag{9}$$

Suppose F_n converged uniformly to F on K. Then $F \in \tilde{V}$ and (9) holds with F in place of F_n, that is,

$$\chi_{j'}(F) = (-1)^j \chi_{j'}(f). \tag{10}$$

Since $\chi_{j'}$ converges to χ and $\chi(f) = 1$, we have

$$\chi_{j'}(F) = (-1)^j \chi_{j'}(f) \approx (-1)^j, \tag{11}$$

for sufficiently large j, that is, $\chi_{j'}$ does not converge.

It remains to show that F_n converges uniformly. If F_n converged uniformly *except* on arbitrarily small neighborhoods of the "slice" $\{y\} \times K_2$, then we could conclude that the F_n converged uniformly on K. The following argument shows how to *modify* f so that (9)–(11) hold and so that the new F_n converge uniformly. The function $z \to f(y, z)$ is continuous on K_2. Let w be this function, that is, $w(z') = f(y, z')$. Then $\| w \|_\infty \leqslant 1$. Let $f' = f - 1 \otimes w$. A moment's thought shows that f' is the necessary modification of f. This completes the proof of Theorem 3.

4. Proof of Theorem 4

Let $x = (y, z)$. If either y or z is isolated in its factor of K, then $V(K)$ restricted to the closed neighborhood $\{y\} \times K_2$ (or $K_1 \times \{z\}$, as the case may be) of x is all of $C(\{y\} \times K_2)$ (see Example (ii) above). Hence the fiber over x is one point.

We may therefore assume that neither y nor z is isolated. Then as in [*12*, formulas 6–7] we can find a sequence F_j of elements of $V(K)$ such that support F_j is contained in the closed rectangle $Y_j \times Z_j$ such that

$$Y_j \cap Y_k = \varnothing = Z_j \cap Z_k, \qquad j \neq k; \tag{12}$$

$$\| F_j \|_\infty < 2^{-j-1}; \tag{13}$$

$Y_j \times Z_j$ is contained eventually in each neighborhood of x; and such that for any choice $c_1, ..., c_n$ of complex numbers

$$\sup \left\{ \left\| \sum_{k=1}^{n} c_k F_j{}^k \right\|_V : j = 1, 2, ... \right\} = \sum_{k=1}^{n} | c_k |. \tag{14}$$

We set $F = \sum_{j=1}^{\infty} F_j$. By applying Lemma B of Section 2 repeatedly (and using formulas (12) and (13)) we see that $F \in \tilde{V}$. Formula (14) shows that the spectral radius of $(F + 1)$ is two. Hence there exists a maximal ideal χ of $\tilde{V}$ such that $\chi(F) = 1$.

On the other hand, if μ is any maximal ideal of $\tilde{V}$ and $\mu \notin \pi^{-1}(x)$, then $\mu(F)$ is given by

$$\mu\left(\sum_{j=1}^{N} F_j\right), \tag{15}$$

for some finite N. This follows from the fact that the $Y_j \times Z_j$ are eventually in a neighborhood of x which does not contain $\pi(\mu)$. Hence by (13), and the fact that $F_j \in V$ are constant on fibers, we have $|\mu(F)| \leqslant \frac{1}{2}$.

Therefore, the non-void open set $\{\chi\colon |\chi(F)| > \frac{3}{4}\}$ is contained in $\pi^{-1}(x)$.

5. Proof of Theorem 5

Definition. A function F defined on $[-1, 1]$ (resp. R) operates on $\tilde{V}(K)$ if for each $f \in \tilde{V}(K)$ with $f(K) \subseteq [-1, 1]$ (resp. $f(K) \subseteq R$) we have $F \circ f \in \tilde{V}(K)$.

The proof of Theorem 5 is merely a matter of using standard arguments concerning functions that operate on Fourier–Stieltjes transforms. From [*12*] we know $\tilde{V}$ contains a function f real on K such that for each choice of complex numbers $c_1, \ldots, c_n$ we have

$$\left\| \sum_{j=1}^{n} c_j f^j \right\|_{\tilde{V}} = \sum_{j=1}^{n} |c_j|.$$

From [*10*, Theorem 9.2.5] (proved in [*11*]) we know that F is analytic in a neighborhood of $[-1, 1]$ (resp. R) if F operates on $\tilde{V}$, since F then operates on V.

We need only prove the following Lemma, and then use the arguments of [7, Section 6.3] to complete the proof of Theorem 5.

Lemma E. *Let F be analytic in a neighborhood of R and periodic on R, and let (the restriction of) F to R operate on $\tilde{V}$. Then for each $f \in \tilde{V}$ with $f(K) \subseteq R$, there exist numbers $\epsilon > 0$, $C > 0$ such that $0 \leqslant a \leqslant \epsilon$ implies $\| F(f + a1)\| < C$, where 1 is the function constantly one on K.*

Proof of the Lemma. Since the Theorem is hardly surprising, and since the arguments are standard, we merely give an outline of the steps of the proof of the Lemma.

(a) We may assume that $K = K_1 \times K_2$, where $K_j = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $j = 1, 2$, since $\tilde{V}$ of this K is easily seen to be a restriction of any other $\tilde{V}$, and since F operating on $\tilde{V}(K)$ implies F operates on any restriction of $\tilde{V}(K)$.

(b) There exists a neighborhood U of $x = (0, 0)$ and numbers ϵ', $C' > 0$ such that $g \in \tilde{V}$, $g(K) \subseteq R$, $\| g \| < \epsilon'$ and support $g \subseteq U \setminus \{(0, 0)\}$ imply $\| F(f + g) \| \leqslant C'$.

(c) If W is any rectangular neighborhood of x, then there exist constants $\epsilon(W)$, $C(W) > 0$ such that for any $g \in \tilde{V}$ having $g(K) \subset R$ and $\| g \| \leqslant \epsilon(W)$, we have

$$\| F(f + g)|_{K \setminus W} \| < C(W).$$

This follows from Gelfand's Cauchy formula for symbolic calculus [*1*, p. 20. Formula (1)] and that $y \to y^{-1}$ is norm continuous; we need only observe that $K \setminus W$ is the maximal ideal space of $\tilde{V}$ restricted to $K \setminus W$, and that this restriction is $C(K \setminus W)$. This last follows from the fact that $V(K)$ restricted to $K \setminus W$ is $C(K \setminus W)$; and this from breaking $K \setminus W$ into the union of three sets, one of which is the product of two finite sets, and the other two the product of a finite with an infinite set (see Examples (i) and (ii) above).

(d) Putting (b) and (c) together, we see that if $g \in V$, $g^2 = g$, $\| g \| \leqslant 2$ and $g = 0$ in a neighborhood of x, then for

$$0 \leqslant a \leqslant 1/2 \inf(\epsilon', \epsilon(U)),$$

we have the function

$$G(g, a) = F(gf + ag) + (1 - g) F(f(x) + a)$$

which has norm

$$\| G(g, a) \| \leqslant C' + C(U) + 3 \| F \|_\infty = C,$$

where $\| F \|_\infty$ is the supremum of $| F(r) |$, $r \in R$. Since F is periodic and continuous, this supremum is finite.

(e) Now let g run through a sequence g_n of elements of V which satisfy (d), and such that the neighborhoods $g_n^{-1}(0)$ of x tend to $\{x\}$. Then the functions $G(g_n, a)$ tend uniformly to $F(f + a\,1)$. Hence $\| F(f + a\,1) \| \leqslant C$ when $0 \leqslant a \leqslant \epsilon$, where $\epsilon = \frac{1}{2} \inf(\epsilon', \epsilon(U))$.

6. Unanswered Questions

(1) What are the elements $f \in V$ such that there exist $g_j \in C(K_1)$ and $h_j \in C(K_2)$ such that

$$\|f\| = \sum \|g_j\|_\infty \|h_j\|_\infty \ ?$$

(2) Is the embedding of V in $\tilde{V}$ in always an isometry (see [*12*])?

(3) What is the Silov boundary for $\tilde{V}$? What is the Silov boundary for $\tilde{V}/I$, where I is as in the proof of Theorem 2? Is $\tilde{V}/I$ uniformly closed in $C(\pi^{-1}(x))$?

(4) What are the homeomorphisms of K with itself which map $\tilde{V}$ onto $\tilde{V}$?

(5) Suppose K_j is homeomorphic to the circle, for $j = 1, 2$. Let G_j be the group of homeomorphisms of K_j with itself which leave $1 \in K_j$ fixed. Then $G_1 \times G_2$ operates on the maximal ideal space of $\tilde{V}(K_1 \times K_2)$ and the fiber $\pi^{-1}(1, 1)$ is mapped onto itself by each element of $G_1 \times G_2$. Is $G_1 \times G_2$ transitive on $\pi^{-1}(1, 1) \setminus \{(1, 1)\}$? Same question for K_j convergent sequences and G_j the permutations of the nonlimit points.

(6) Using the function g constructed in the proof of Theorem 1, we see that a nontrivial fiber is not a set of spectral synthesis [*10*, p. 56] for $\tilde{V}$. The same construction shows that a cylinder $\pi^{-1}(S)$ is "rarely" a set of spectral synthesis for $\tilde{V}$. Are there any nontrivial sets of spectral synthesis for $\tilde{V}$?

(7) Let $K = K_1 \times K_2$ and $K' = K_1' \times K_2'$. Let V, $\tilde{V}$, π and V', $\tilde{V}'$ and π' be the corresponding algebras and projections. Let $x \in K$ and $x' \in K'$. Suppose the fibers $\pi^{-1}(x)$ and $\pi'^{-1}(x')$ are nontrivial. Are they then homeomorphic? Are they homeomorphic if the K_j and $K_{j'}$ are totally disconnected?

(8) The corollary to Theorem 1 shows that V is not characterized as a maximal subalgebra of $\tilde{V}$ with K as its maximal ideal space. Is V the maximal subalgebra of $\tilde{V}$ such that every point of K is a set of spectral synthesis for the algebra (see Question (6) above)?

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